The Action Principle in Sourceless Classical Electrodynamics

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ABSTRACT

This summary presents a modern formulation of classical electromagnetism grounded in the action principle. Beginning with the Lagrangian density that incorporates the electromagnetic field strength tensor, we introduce the action formalism in the context of field theory. The equations of motion for the electromagnetic field are Maxwell's equations. Two of these can be directly derived from the principle of least action applied to the Lagrangian, offering an elegant and natural connection between variational principles and physical laws. The remaining two equations are automatically satisfied as a consequence of the definition of the electromagnetic field tensor.

The symmetries of the action are examined through Noether's theorem, which demonstrates that conservation of energy and momentum arises from the invariance of the action under spacetime translations. We proceed to derive the electromagnetic energy–momentum tensor and interpret its physical components, including the energy density and the Poynting vector. This approach unifies the dynamical equations and conservation laws of electromagnetism within a coherent Lagrangian framework, effectively bridging classical electrodynamics and modern theoretical physics.

Keywords: Classical Electromagnetism, Action Principle, Lagrangian Density, Field Theory, Maxwell's Equations, Euler–Lagrange Equations, Noether's Theorem, Energy–Momentum Tensor, Poynting Vector.

I. NOTATIONS AND CONVENTIONS

- We work in natural units, where $\varepsilon_0 = \mu_0 = c = 1$.
- The metric tensor is taken to be of the form diag$(+, -, -, -)$.
- Greek indices $(\mu, \nu, \dots)$ run over the spacetime coordinates $0, 1, 2, 3$, corresponding to (t, x, y, z).
- Roman indices $(i, j, \dots)$ denote the spatial components $1, 2, 3$.
- We employ the Einstein summation convention: repeated indices, one upper and one lower, are implicitly summed over.

II. PRELIMINARY: THE ACTION PRINCIPLE

The action principle lies at the heart of modern theoretical physics and has its roots in the work of pioneering figures such as Pierre Maupertuis and Leonhard Euler in the 18th century, later refined by William Rowan Hamilton and Carl Jacobi. It offers a profound and elegant framework in which the dynamics of both particles and fields can be derived from a single scalar quantity: the action. By demanding that the action be stationary under small variations—a concept known as Hamilton's principle—one can obtain the equations of motion for a wide range of physical systems. This variational approach not only unifies classical mechanics, electromagnetism, general relativity, and quantum field theory under a common formalism but also naturally encodes symmetries and conservation laws through Noether's theorem. As such, the action principle stands as one of the most powerful and unifying concepts in all of physics.

A. Action and Lagrangian

The action is defined as the integral of a function known as the Lagrangian over time:

$$S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt,$$

where $q(t)$ represents the generalized coordinates describing the configuration of the system, $\dot{q}$ denotes their time derivatives, and t is time. The Lagrangian L is chosen to reflect the physical properties of the system, typically depending on position, velocity, and possibly time explicitly.

The action S plays a foundational role in determining the evolution of a system. Among all possible paths that a system could take between two fixed configurations, only one corresponds to the actual physical motion. This path is determined by requiring that the action be stationary under small variations, a condition known as the principle of stationary action.

In extending this formalism from particles to fields, the idea of the action is generalized naturally. Instead of functions of time alone, we consider quantities that vary over spacetime. In field theory, the dynamical object is no longer a set of coordinates $q(t)$, but a field $\phi(x^\mu)$, where $x^\mu = (t, x, y, z)$ labels points in spacetime. The Lagrangian is replaced by a Lagrangian density $\mathcal{L}$, and the action becomes an integral over four-dimensional spacetime:

$$\boxed{S[\phi] = \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\, d^4x}$$

This formulation allows for a consistent description of systems with infinitely many degrees of freedom—such as electromagnetic or quantum fields—while preserving the elegance and generality of the variational principle.

B. Equations of Motion: Euler–Lagrange Equations

To determine the dynamics of the field, we apply the principle of stationary action: the physical configuration of the field is the one that makes the action $S[\phi]$ stationary under small variations $\delta\phi(x^\mu)$, subject to the condition that the variation vanishes at the boundaries of the spacetime region under consideration:

$$\delta S = 0 \quad \text{subject to the boundary condition } \delta\phi(x^\mu)\big|_{\text{boundary}} = 0,$$

We compute the variation of the action:

$$\begin{aligned}
\delta S &= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] d^4x \\
&= \int \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi\, d^4x + \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi)\, d^4x \\
&= \int \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi\, d^4x + \left[\int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) d^4x - \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi\, d^4x \right] \\
&= \int \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi\, d^4x - \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi\, d^4x \\
&= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi\, d^4x.
\end{aligned}$$

For the action to be stationary for arbitrary variations $\delta\phi$ (that vanish at the boundary), the integrand must vanish identically:

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

These are the Euler–Lagrange equations for classical fields. They provide the equations of motion governing the dynamics of the field $\phi(x^\mu)$, derived directly from the Lagrangian density via the variational principle.

C. Noether's Theorem and Conservation Laws

Noether's theorem provides a powerful and elegant method for deriving conservation laws in classical field theory from continuous symmetries of the action. It plays a central role in modern theoretical physics, revealing the deep relationship between invariance principles and conserved quantities.

Let us consider a general infinitesimal transformation of the fields of the form:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x),$$

where α is an infinitesimal parameter and $\Delta\phi(x)$ is some function of the field and its derivatives. This transformation is said to be a symmetry of the theory if the action remains invariant under the transformation, or more generally, if the change in the action is a boundary term that does not affect the equations of motion.

The corresponding change in the Lagrangian density is:

$$\begin{aligned}\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\Delta\phi) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right)\end{aligned}$$

Now suppose that the transformation is a symmetry of the theory. In that case, the change in the Lagrangian must also be expressible as a total derivative:

$$\Delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu,$$

for some function $\mathcal{J}^\mu$. Equating the two expressions for $\Delta\mathcal{L}$, we find:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu\right) = 0.$$

This implies that the quantity:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu$$

is a conserved current, satisfying the continuity equation:

$$\partial_\mu j^\mu = 0.$$

This is precisely Noether's theorem: for every continuous symmetry of the action, there exists a corresponding conserved current. Let us apply this result to the case of spacetime translations. Consider an infinitesimal translation:

$$x^\mu \to x'^\mu = x^\mu - a^\mu,$$

where a^μ is a constant infinitesimal vector. Under this transformation, the field changes as:

$$\phi(x) \to \phi'(x) = \phi(x+a) = \phi(x) + a^\nu\partial_\nu\phi(x),$$

so that:

$$\Delta\phi = a^\nu\partial_\nu\phi.$$

Similarly, the Lagrangian transforms as:

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\nu\partial_\nu\mathcal{L}(x),$$

which implies:

$$\Delta\mathcal{L} = a^\nu\partial_\nu\mathcal{L} = \partial_\mu(a^\nu\delta^\mu{}_\nu\mathcal{L}) = \partial_\mu(a^\mu\mathcal{L}) = \partial_\mu\mathcal{J}^\mu,$$

Substituting into the expression for the Noether current:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu = a^\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \mathcal{L}\delta^\mu{}_\nu\right).$$

We define the energy-momentum tensor $T^\mu{}_\nu$ by:

$$\boxed{T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \mathcal{L}\delta^\mu{}_\nu}$$

Then the current becomes:

$$j^\mu = a^\nu T^\mu{}_\nu.$$

Since $\partial_\mu j^\mu = 0$, and a^ν is arbitrary, we conclude that:

$$\partial_\mu T^\mu{}_\nu = 0.$$

This is the local conservation law for energy and momentum. The tensor $T^{\mu\nu}$ encodes the energy density, momentum density, and stress in the field, and its conservation is a direct consequence of the invariance of the action under spacetime translations.

III. PRELIMINARY: CLASSICAL ELECTRODYNAMICS

Classical electrodynamics is a fundamental field theory that describes the dynamics of electric and magnetic fields. In the absence of charges and currents the electromagnetic field evolves autonomously, governed entirely by its internal structure and conservation laws. We review Maxwell's equations in vacuum and recap how energy is stored and transported in the electromagnetic field.

A. Maxwell's Equations in Vacuum

The four fundamental equations of classical electrodynamics in vacuum are:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} \end{aligned}}$$

We work in natural units, where $\varepsilon_0 = \mu_0 = 1$. These constants are absorbed into the definitions of the fields, simplifying the form of the equations while preserving all physical content.

Each of these equations has a distinct origin and interpretation:

- The equation $\nabla \cdot \mathbf{E} = 0$ is Gauss's law in vacuum with no electric charges. Electric field lines have no sources or sinks and must form closed loops or extend to infinity.
- The equation $\nabla \cdot \mathbf{B} = 0$ is the magnetic version of Gauss's law. It reflects the absence of magnetic monopoles: magnetic field lines are continuous and never begin or end.
- The equation $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ is Faraday's law. It states that a changing magnetic field induces a circulating electric field.
- The equation $\nabla \times \mathbf{B} = \partial_t \mathbf{E}$ is Ampère's law with Maxwell's correction. It shows that a changing electric field generates a magnetic field. $\partial_t \mathbf{E}$ is the displacement current, added by Maxwell to ensure consistency with charge conservation.

Together, these equations describe a fully consistent, dynamically closed system for the electric and magnetic fields in vacuum. They predict the existence of self-sustaining electromagnetic waves, although we will not explore their derivation here.

B. Energy Density and the Poynting Vector

Even in the absence of sources, the electromagnetic field itself carries energy and transports it through space. Two key quantities describe this aspect of the field:

- The energy density, denoted $\mathcal{E}$, represents the amount of energy per unit volume stored in the electromagnetic field.
- The Poynting vector, denoted $\mathbf{S}$, gives the directional flow of electromagnetic energy.

They are defined as:

$$\boxed{\begin{aligned} &\mathcal{E} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \\ &\mathbf{S} = \mathbf{E} \times \mathbf{B}. \end{aligned}}$$

To understand the physical meaning of these expressions, consider the time derivative of the energy density:

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) \end{aligned}$$

Applying a standard vector identity and rearranging terms yields:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{S} = 0.$$

This is the Poynting theorem in vacuum. It expresses the local conservation of electromagnetic energy: the rate of change of energy density in a region is balanced by the net outward flux of the Poynting vector through the boundary of that region.

IV. THE ACTION PRINCIPLE IN SOURCELESS CLASSICAL ELECTRODYNAMICS

Here, we apply the variational principle to sourceless (vacuum) electromagnetism. We begin by deriving Maxwell's equations from an action built out of the electromagnetic field strength tensor. Then, we construct the energy–momentum tensor of the electromagnetic field and show how it leads naturally to expressions for energy density and the Poynting vector.

A. Deriving Maxwell's Equations from the Action

In the source-free case, the dynamics of the electromagnetic field are governed by the vector potential $A_\mu(x)$, a four-vector field whose derivatives define the electromagnetic field strength:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

This antisymmetric tensor $F_{\mu\nu}$ contains all components of the electric and magnetic fields and transforms covariantly under Lorentz transformations, ensuring relativistic invariance.

The simplest scalar we can construct from $F_{\mu\nu}$ is the contraction $F_{\mu\nu}F^{\mu\nu}$. This leads naturally to the following action for the electromagnetic field:

$$\boxed{S[A] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)}$$

This form of the action is chosen because it is:

- Lorentz-invariant. It's built from a scalar formed from tensors,
- Local. It depends only on fields and their first derivatives at each spacetime point,
- Hindsight. The factor of $-1/4$ is added so it leads to the results of classical electrodynamics, as we shall see.

The Lagrangian of the system is therefore:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

We apply the Euler-Lagrange equations. Note that the Lagrangian contains no explicit dependence on A_β.

$$\begin{aligned}
0 &= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)}\right) - \frac{\partial\mathcal{L}}{\partial A_\beta} \\
&= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)}\right) \\
&= -\frac{1}{4}\partial_\alpha\left(\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\right]\right) \\
&= -\frac{1}{4}\eta^{\mu\gamma}\eta^{\nu\delta}\partial_\alpha\left(\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\gamma A_\delta - \partial_\delta A_\gamma)\right]\right) \\
&= -\frac{1}{4}\eta^{\mu\gamma}\eta^{\nu\delta}\partial_\alpha\left((\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{\partial}{\partial(\partial_\alpha A_\beta)}(\partial_\gamma A_\delta - \partial_\delta A_\gamma) + (\partial_\gamma A_\delta - \partial_\delta A_\gamma)\frac{\partial}{\partial(\partial_\alpha A_\beta)}(\partial_\mu A_\nu - \partial_\nu A_\mu)\right) \\
&= -\frac{1}{4}\eta^{\mu\gamma}\eta^{\nu\delta}\partial_\alpha\left((\partial_\mu A_\nu - \partial_\nu A_\mu)(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma) + (\partial_\gamma A_\delta - \partial_\delta A_\gamma)(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)\right) \\
&= -\frac{1}{4}\partial_\alpha\left((\partial_\mu A_\nu - \partial_\nu A_\mu)(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}) + (\partial_\gamma A_\delta - \partial_\delta A_\gamma)(\eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\beta\gamma}\eta^{\alpha\delta})\right) \\
&= -\partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \\
&= -\partial_\alpha F^{\alpha\beta}
\end{aligned}$$

Relabeling indicies, we obtain:

$$\boxed{\partial_\nu F^{\mu\nu} = 0}$$

Note that μ is the free index in this expression. In addition, we make use of the antisymmetry property of the field tensor ($F^{\mu\nu} = 0, \text{ if } \mu = \nu$).

- Case $\mu = 0$:

$$0 = \partial_\nu F^{0\nu} = \partial_j F^{0j} = -\partial_j E^j = -\nabla\cdot\mathbf{E} \implies \boxed{\nabla\cdot\mathbf{E} = 0}$$

- Case $\mu = i$:

$$\begin{aligned} 0 &= \partial_\nu F^{i\nu} \\ &= \partial_0 F^{i0} + \partial_j F^{ij} \\ &= \partial_0 E^i - \varepsilon^{ijk}\partial_j B_k \\ &= \partial_t E^i - (\nabla \times \mathbf{B})^i \implies \boxed{\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}} \end{aligned}$$

The variational principle gives us one tensor equation, which corresponds to two of Maxwell's equations when decomposed into spatial and temporal components. The other two equations arise not from the action, but from the definition of the field strength tensor and the associated Bianchi identity.

$$\boxed{\partial_\lambda F_{\mu\nu} \;+\; \partial_\mu F_{\nu\lambda} \;+\; \partial_\nu F_{\lambda\mu} \;=\; 0.}$$

$F_{\mu\nu}$ is antisymmetric, so this cyclic sum vanishes identically. Specialising to particular choices of indices yields the remaining two Maxwell equations.

B. Energy–Momentum Tensor, Energy Density, and Poynting Vector

As mentioned in the preliminary section, the energy-momentum tensor is defined as:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\gamma)}\partial_\nu A_\gamma - \delta^\mu_\nu \mathcal{L}$$

And we have calculated already that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\gamma)} = -F^{\mu\gamma}$$

We first obtain the energy–momentum tensor in mixed form, and then raise its indices to express it in fully contravariant form.

$$\begin{aligned} T^\mu{}_\nu &= -F^{\mu\gamma}\partial_\nu A_\gamma + \frac{1}{4}\delta^\mu{}_\nu F_{\alpha\beta}F^{\alpha\beta} \\ T^{\mu\nu} &= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} \end{aligned}$$

However, this energy–momentum tensor is not symmetric: $T^{\mu\nu} \neq T^{\nu\mu}$. The asymmetry comes from the term $-F^{\mu\gamma}\partial^\nu A_\gamma$, which is not invariant under exchanging μ and ν due to the vector nature of A_μ.

Symmetry is essential because the energy–momentum tensor couples to the symmetric metric in general relativity, and it must also ensure conservation of angular momentum. To fix this, we add a term $\partial_\lambda(F^{\mu\lambda}A^\nu)$. This term is antisymmetric, and therefore is automatically divergenceless.

$$\begin{aligned} \hat{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu) \\ &= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{\mu\lambda})A^\nu + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= -F^{\mu\gamma}\partial^\nu A_\gamma + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= -F^{\mu\lambda}\partial^\nu A_\lambda + F^{\mu\lambda}\partial_\lambda A^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ &= \boxed{-F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}} \end{aligned}$$

This tensor is symmetric, as the product of two electromagnetic field tensor (which is antisymmetric) will result in a symmetric tensor. Most importantly, this is an equally valid energy-momentum tensor, yielding the same globally conserved energy and momentum. Let us now examine its energy density, given by $\mathcal{E} = T^{00}$, as well as the Poynting vector, which represents the momentum density.

For ease of calculation, we lower the index for the mixed tensor.

$$\boxed{\hat{T}^{\mu\nu} = \eta^{\rho\nu}F^{\mu\lambda}F_{\lambda\rho} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}}$$

Recall that the components of the electromagnetic field tensor are given by:

$$\boxed{F^{0i} = -E^i, \quad F^{ij} = -\varepsilon^{ijk}B_k}$$

Note that lowering the indexes for the field tensor may yield an additional negative sign. It is important here to distinguish between 3-tensors and 4-tensors. Objects in 3D space (electric field and magnetic field) use the Euclidean "+++" metric, so their covariant and contravariant components are identical. In contrast, tensors in 4D spacetime (like the electromagnetic tensor) use the "+—" metric, so lowering any spatial index introduces a negative sign.

$$F_{\sigma\rho}\Big|_{\sigma=0,\,\rho=i} = \eta_{\sigma 0}\eta_{\rho i}F^{0i}\Big|_{\sigma=0,\,\rho=i} \implies F_{0i} = -F^{0i}$$

$$E^i = E_i$$

$$\boxed{F_{0i} = E_i}$$

And similarly,

$$F_{\sigma\rho}\Big|_{\sigma=i,\,\rho=j} = \eta_{\sigma i}\eta_{\rho j}F^{ij}\Big|_{\sigma=i,\,\rho=j} \implies F_{ij} = F^{ij}$$

$$\varepsilon^{ijk}B_k = \varepsilon_{ijk}B^k$$

$$\boxed{F_{ij} = -\varepsilon_{ijk}B^k}$$

We begin by computing the energy density of the field, corresponding to the component $\mu = \nu = 0$:

$$\begin{aligned}
\mathcal{E} &= \hat{T}^{00} \\
&= \eta^{\rho 0}F^{0\lambda}F_{\lambda\rho} + \frac{1}{4}\eta^{00}F_{\alpha\beta}F^{\alpha\beta} \\
&= F^{0i}F_{i0} + \frac{1}{4}(F_{i0}F^{i0} + F_{ij}F^{ij} + F_{0j}F^{0j}) \\
&= (-E^i)(-E_i) + \frac{1}{4}[-E_iE^i + (-\varepsilon_{ijk}B^k)(-\varepsilon^{ijl}B_l) + (-E_j)(E^j)] \\
&= \frac{1}{2}E^iE_i + \frac{1}{4}(2\delta^l_k)B^kB_l \\
&= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)
\end{aligned}$$

Next, we calculate the Poynting vector, corresponding to the component $\mu = 0,\, \nu = m$:

$$\begin{aligned}
\mathbf{S}^m &= \hat{T}^{0m} \\
&= \eta^{\rho m}F^{0\lambda}F_{\lambda\rho} + \frac{1}{4}\eta^{0m}F_{\alpha\beta}F^{\alpha\beta} \\
&= -F^{0i}F_{im} \\
&= -\varepsilon_{imk}E^iB^k \\
&= (\mathbf{E}\times\mathbf{B})^m
\end{aligned}$$

These results match the expressions for the electromagnetic field energy density and Poynting vector given in the previous section. We have therefore successfully formulated a description of electromagnetism starting from the action principle.